KNOT HOMOTOPY IN SUBSPACES OF THE 3-SPHERE

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Abstract. We discuss an "extrinsic" property of knots in a 3-subspace of the 3-sphere S^3 to characterize how the subspace is embedded in S^3. Specifically, we show that every knot in a subspace of the 3-sphere is transient if and only if the exterior of the subspace is a disjoint union of handlebodies, i.e. regular neighborhoods of embedded graphs, where a knot in a 3-subspace of S^3 is said to be transient if it can be moved by a homotopy within the subspace to the trivial knot in S^3. To show this, we discuss relation between certain group-theoretic and homotopic properties of knots in a compact 3-manifold, which can be of independent interest. Further, using the notion of transient knot, we define an integer-valued invariant of knots in S^3 that we call the transient number. We then show that the union of the sets of knots of unknotting number one and tunnel number one is a proper subset of the set of knots of transient number one.

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Introduction

In [12] Fox proposed a program to distinguish 3-manifolds by the differences in their "knot theories". Following the program, Brody [6] re-obtained the topological classification of the 3-dimensional lens spaces using knot-theoretic invariants, which are the Alexander polynomials of knots suitably factored out so that it depends only on the homology classes of the knots. Bing's recognition theorem [2] can be regarded as another example of works that follow Fox's program, where the theorem asserts that a closed, connected 3-manifold M is homeomorphic to the 3-sphere if and only if every knot in M can be moved by an isotopy to lie within a 3-ball. We note here that if we replace *isotopy* in this statement by *homotopy*, the assertion implies Poincaré Conjecture, which was proved by Perelman [34, 35, 36]. Bing's recognition theorem was generalized by Hass-Thompson [16] and Kobayashi-Nishi [22] proving that a closed, connected 3-manifold M admits a genus g Heegaard splitting if and only if there exists a genus g handlebody V embedded in M so that every knot in M can be moved by an isotopy to lie within V. We note that as mentioned in Nakamura [30], the *homotopy* version of this statement holds when $g = 1$ again by Poincaré Conjecture whereas that for higher genus case fails in general. A result of Brin-Johannson-Scott [4] can also be regarded as a work following Fox's program, which asserts that if every knot in M can be moved by a homotopy to lie within a collar neighborhood of the boundary ∂M, then there

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exists a component F of ∂M such that the natural map $\pi_1(F) \to \pi_1(M)$ induced by the inclusion is surjective. In particular, for a compact, connected, orientable, irreducible, boundary-irreducible 3-manifold M, they proved that if every knot in M can be moved by a homotopy to lie within a collar neighborhood of ∂M, then M is homeomorphic to the 3-ball or the product $\Sigma \times [0,1]$, where Σ is a closed, orientable surface of genus at least one. In the present paper, we will consider a relative version of Fox's program, namely, we discuss "(extrinsic) knot theories" in 3-subspaces of the 3-sphere S^3 in order to characterize how the 3-subspaces are embedded in S^3.

Let M be a compact, connected, proper 3-submanifold of S^3. We say that M is *unknotted* if its exterior is a disjoint union of handlebodies. A famous theorem of Fox [13] says that each M can be re-embedded in S^3 so that its image is unknotted. A re-embedding satisfying this property are called a *Fox re-embedding*. Intuitively speaking, unknottedness of $M \subset S^3$ implies that M is embedded S^3 in one of the "simplest" ways. We note that if M is a handlebody, an unknotted M in S^3 is actually unique up to isotopy by Waldhausen [46]. The uniqueness up to isotopy and a reflection holds for each knot exterior by a celebrated result of Gordon-Luecke [15]. However, in other cases M usually admits many mutually non-isotopic Fox re-embeddings into S^3.

The unknottedness of a 3-submanifold and so the existence of a Fox re-embedding can be considered for an arbitrary closed, connected 3-manifold. Scharlemann-Thompson [41] generalized the above theorem of Fox by proving that any compact, connected, proper 3-submanifold of irreducible non-Haken 3-manifolds N admits a Fox re-embedding into N or S^3. Another generalization is given by Nakamura [30] who proved that a compact, connected, proper 3-submanifold M of a closed, connected 3-manifold N admits a Fox re-embedding into N if every knot in N can be moved by an isotopy to lie within M. Here we remark that the property that

every knot in N can be moved by an isotopy to lie within M

does *not* imply that M itself is unknotted in N. This can be seen for example by considering the case where $N = S^3$ and M is not unknotted. In this paper, we will show that the property of a compact, connected, proper 3-submanifold M of S^3 that

every knot in M can be moved by a homotopy in M to be the trivial knot in S^3

implies that M is unknotted in S^3. Following Letscher [26], we say that a knot K in M is *transient in M* if K can be deformed by a homotopy in M to be the trivial knot in S^3; K is said to be *persistent in M* otherwise. Using this terminology, we can state our main theorem as follows:

Theorem 3.2. *Let M be a compact, connected, proper 3-submanifold of S^3. Then every knot in M is transient in M if and only if M is unknotted.*

Roughly speaking, the above theorem implies that a (homotopic) property of knots in M deduces an isotopic property of M inside S^3. We remark that the property that a given knot $K \subset M$ is transient is *extrinsic* with respect to the embedding $M \hookrightarrow S^3$ in the sense that it depends not only the pair (M, K) but also the way how M is embedded in S^3. Indeed, we can find a persistent knot in a certain genus two handlebody V embedded in S^3 in such a way that there exists another embedding of V into S^3 such that the re-embedded knots in the re-embedded V is transient. See Section 3. Now, we

can say a little more precisely what is the relative version of Fox's program; we expect that extrinsic properties for knots in a compact, connected, proper 3-submanifold of S^3 distinguish the isotopy class of M inside S^3. Our main theorem is a first step for the program. To obtain the theorem, we discuss relation between certain group-theoretic and homotopic properties of knots in a compact 3-manifold, which can be of independent interest, see Section 1.

Given a knot K in a compact, connected, proper 3-submanifold M of S^3, it is actually difficult in general to detect if K is persistent in M. One method was provided by Letscher [26] that uses what he calls the *persistent Alexander polynomial*. In Section 4, we provide examples of persistent knots in a 3-subspace of S^3 whose persistency are shown by using the notion of *persistent lamination* and *accidental surface*.

In Section 5, we will introduce an integer-valued invariant, *transient number*, of knots in S^3 whose definition is related to Theorem 3.2 as follows. Given a knot K in S^3, we may consider a system of simple arcs in S^3 with their endpoints in K such that K is transient in a regular neighborhood of the union of K and the arcs. The transient number $tr(K)$ is then defined to be the minimal number of simple arcs in such a system. By an easy observation, we see that the transient number is bounded from above by both the unknotting number and the tunnel number. Further, we will give a knot K that attains $tr(K) = 1$ while $u(K) = t(K) = 2$, where $u(K)$ and $t(K)$ are the unknotting number and the tunnel number of K, respectively (see Proposition 5.2). In other words, the union of the sets of knots of unknotting number one and tunnel number one is actually a proper subset of the set of knots of transient number one. The final section contains some concluding remarks and open questions.

Throughout this paper, we will work in the piecewise linear category.

Notation. Let X be a subset of a given polyhedral space Y. Throughout the paper, we will denote the interior of X by $\operatorname{Int} X$. We will use $\operatorname{Nbd}(X;Y)$ to denote a closed regular neighborhood of X in Y. If the ambient space Y is clear from the context, we denote it briefly by $\operatorname{Nbd}(X)$. Let M be a 3-manifold. Let $L \subset M$ be a submanifold with or without boundary. When L is 1 or 2-dimensional, we write $E(L) = M \setminus \operatorname{Int}\operatorname{Nbd}(L)$. When L is of 3-dimension, we write $E(L) = M \setminus \operatorname{Int} L$. We shall often say surfaces, compression bodies, e.t.c. in an ambient manifold to mean the isotopy classes of them.

1. Knots filling up a handlebody

Let F_g be a free group of rank g with a basis $X_g = \{x_1, x_2, \dots, x_g\}$. We set $X_g^{\pm} = X_g \cup \{{x_1}^{-1}, {x_2}^{-1}, \dots, {x_g}^{-1}\}$. A *word* on X_g is a finite sequence of letters of $X_g^{\pm}$. For an element x of a group G, we denote by $c_G(x)$ (or simply by $c(x)$) its conjugacy class in G.

Let G be a group with a decomposition $G = G_1 * G_2$. Then G_1 and G_2 are called *free factors* of G. In particular, if $G_2 \neq 1$, then G_1 is called a *proper* free factor of G. Following Lyon [27], we say that an element x of G *binds* G if x is not contained in any proper free factor of G. Thus, for example, an element of $\mathbb{Z}$ binds $\mathbb{Z}$ if and only if it is non-trivial. We can also see that an element of a rank 2 free group $F_2 = \langle x_1, x_2 \rangle$ binds F_2 if and only if it is not a power of primitive element, where an element of a free group

is said to be *primitive* if it is a member of some of its free basis. For example $x_1x_2x_1x_2$ does not bind F_2 while $x_1x_2x_1{x_2}^3$ binds F. See e.g. Osborne-Zieschang [32] and Cho-Koda [8]. Primitive elements of the rank 2 free group have been well understood by e.g. Osborne-Zieschang [32] and Cohen-Metzler-Zimmermann [10] whereas their classification in a free group of higher rank is known to be a hard problem. See Puder-Wu [38] (and also Shpilrain [42]) and Puder-Parzanchevski [37] for some of the deepest results on this problem. On the contrary, an algorithm to detect if a given element x of a free group F_g binds F_g is given by Stallings [43] using the combinatorics of its Whitehead graph. See Section 6 (2). It follows immediately from the definition that if x binds G, then any element of its conjugacy class $c(x)$ binds G. In fact, if $x \in G_1$ for a decomposition $G = G_1 * G_2$, then $a^{-1}xa \in a^{-1}G_1a$ and $F = (a^{-1}G_1a)*(a^{-1}G_2a)$ is also a decomposition of G for any $a \in G$.

Let K be an oriented knot in a 3-manifold M. We denote by $c_{\pi_1(M)}(K)$ (or simply by $c(K)$) the conjugacy class in $\pi_1(M)$ defined by the homotopy class of K. Here we recall that two oriented knots K and K' in M are homotopic in M if and only if $c_{\pi_1(M)}(K) = c_{\pi_1(M)}(K')$. We say that K *binds* $\pi_1(M)$ if an element (so every element) of $c(K)$ binds $\pi_1(M)$. It is clear by definition that, if $\bar{K}$ is the knot K with the reversed orientation, K binds $\pi_1(M)$ if and only if so does $\bar{K}$. For this reason, we can say whether or not a knot K binds $\pi_1(M)$ ignoring the orientation of K.

Recall that a (possibly disconnected) surface F in a 3-manifold is said to be *compressible* if

(1) there exists a component of F that bounds a 3-ball in M; or
(2) there exists an embedded disk D in M, called a *compression disk* for F, such that $D \cap F = \partial D$ and ∂D is an essential simple closed curve on F.

Otherwise, F is said to be *incompressible*. A 3-manifold is said to be *irreducible* if it contains no incompressible 2-spheres. A 3-manifold is said to be *boundary-irreducible* if its boundary is incompressible. The following lemma is a generalization of Lyon [27, Corollary 1].

Lemma 1.1. *Let M be a compact, connected, irreducible 3-manifold with non-empty boundary. Let K be an oriented simple closed curves in the boundary of M. Then $\partial M \setminus K$ is incompressible in M if and only if K binds $\pi_1(M)$.*

Proof. We fix an orientation and a base point v of K.

Suppose first that K does not bind $\pi_1(M, v)$. Then there exists a decomposition $\pi_1(M, v) = G_1 * G_2$ with $G_2 \neq 1$ and $[K] \in G_1$. Let X_i be a $K(G_i, 1)$-space, and let p be a point not in $X_1 \cup X_2$. We define $\hat{X}_1$ and $\hat{X}_2$ to be the mapping cylinders of maps from p into X_1 and X_2, respectively. Let X denote the space obtained by identifying the copy of p in $\hat{X}_1$ with that of p in $\hat{X}_2$. By the construction, we have $\pi_1(X) = G_1 * G_2$ and $\pi_2(X_1) = \pi_2(X_2) = 0$. Thus there exists a continuous map $f : M \to X$ satisfying the following properties.

(1) $f(v) = a$,
(2) the induced map $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism with $f_*(G_i) = \pi_1(X_i)$ for $i \in \{1, 2\}$, and
(3) $f^{-1}(a)$ consists of a finite number of compression disks for ∂M.

Here we use the assumption that M is irreducible. We may assume that $|f^{-1}(a) \cap K|$ is minimal among all continuous maps $M \to X$ satisfying (1)–(3). Suppose that $f^{-1}(a) \cap K \neq \emptyset$. Then $f(K)$ is a loop in X with the base point a that can be decomposed as

$$f(K) = \alpha_1 * \alpha_2 * \cdots * \alpha_r,$$

where each α_i lies in $\hat{X}_1$ or $\hat{X}_2$, and α_i, α_{i+1} do not lie in one of $\hat{X}_1$ and $\hat{X}_2$ at the same time. We note that $r > 1$. Suppose that none of $[\alpha_i]$ is trivial in G_1 or G_2. Then $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ is a *reduced sequence*, that is, $[\alpha_i]$ is in G_1 or G_2, and $[\alpha_i]$, $[\alpha_{i+1}]$ do not lie in one of G_1 and G_2 at the same time. On the other hand, $[f(K)]$ lies in G_1 by the assumption. This contradicts the uniqueness of reduced sequences, see Magnus-Karrass-Solitar [28, Theorem 4.1]. Thus at least one of $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ is trivial. Consequently, there exists a subarc α of K such that

- $\alpha \cap f^{-1}(a) = \partial\alpha$,
- $f(\alpha) \subset X$ is a contractible loop, and
- α is essential in ∂M cut off by $\partial f^{-1}(a)$.

Then using a standard technique as in Lyon [27, Theorem 2], f can be deformed by a homotopy to be a continuous map $f' : M \to X$ satisfying the above (1)–(3), and $|f'^{-1}(a) \cap K| < |f^{-1}(a) \cap K|$. This contradicts the minimality of $|f^{-1}(a) \cap K|$. Thus we have $f^{-1}(a) \cap K = \emptyset$. This implies that $\partial M \setminus K$ is compressible in M.

Next suppose that there exists a compression disk D for $\partial M \setminus K$ in M. Suppose that D separates M into two components M_1 and M_2, where K lies in M_1. Then $\pi_1(M)$ can be decomposed as $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, where $[K] \in \pi_1(M_1)$. If $\pi_1(M_2) = 1$, then $M_2 \cong B^3$ by the Poincaré conjecture proved in Perelman [34, 35, 36]. This is a contradiction. Hence $\pi_1(M_2) \neq 1$, which implies that K does not bind $\pi_1(M)$. Suppose that D does not separate M. Let M' be M cut off by D. Then we have $\pi_1(M) = \pi_1(M') * \mathbb{Z}$ and $[K]$ is in $\pi_1(M')$. Hence, again, K does not bind $\pi_1(M)$. □

Let M be a compact connected 3-manifold. Let K and K' be knots in M. We denote by $K \overset{M}{\sim} K'$ if K and K' are homotopic in M. Let K be a knot in the interior of M. We say that K *fills* up M if for any knot K' in the interior of M such that $K \overset{M}{\sim} K'$, $E(K')$ is irreducible and boundary-irreducible.

Example. The knot K_1 shown on the left-hand side in Figure 1 does not fill up the handlebody V (becaue there exists a compression disk D for ∂V in $V \setminus K_1$ as shown in the figure) while the knot K_2 shown on the right-hand side fills up V (cf. Lemma 1.5).

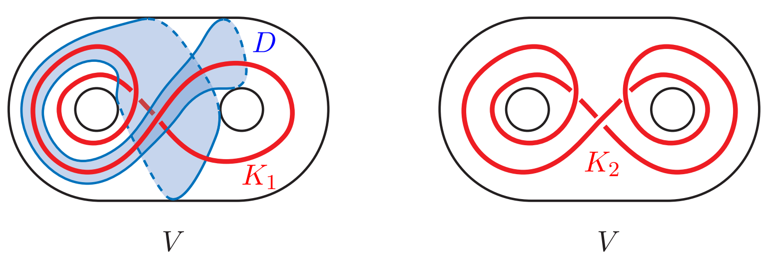

Figure 1. The knot K_1 does not fill up V while K_2 fills up V.

By a *graph*, we mean the underlying space of a (possibly disconnected) finite 1-dimensional simplicial complex. A handlebody is a 3-manifold homeomorphic to a closed regular neighborhood of a connected graph embedded in the 3-sphere. The *genus* of a handlebody is defined to be the genus of its boundary surface. For a handlebody V, a *spine* is defined to be a graph Γ embedded in V so that V collapses onto Γ. By a 1-*vertex spine* we mean a spine with a single vertex. In other words, a 1-vertex spine is a spine of a handlebody that is homeomorphic to a *rose*, i.e. a wedge of circles.

In the remaining of the section we fix the following:

- A handlebody V of genus g at least 1 with a base point v_0.
- A 1-vertex spine Γ_0 of V having the vertex at v_0.
- A standard basis $X = \{x_1, x_2, \ldots, x_g\}$ of $\pi_1(\Gamma_0, v_0) \cong \pi_1(V, v_0)$, that is, we can assign names e^0_1, $e^0_2, \ldots, e^0_g$ and orientations to the edges of Γ_0 so that x_i corresponds to the oriented edge e^0_i for each $i \in \{1, 2, \ldots, g\}$.

Under the above setting, we identify $\pi_1(V) = \pi_1(V, v_0)$ with the free group F with the basis X.

Let $\{y_1, y_2, \ldots, y_g\}$ be a basis of F, where each y_i is a word on the standard basis X. We say that a 1-vertex spine Γ of V having the vertex at v_0 is *compatible with* the basis $\{y_1, y_2, \ldots, y_g\}$ if we can assign names e_1, $e_2, \ldots, e_g$ and orientations to the edges of Γ so that a word on X corresponding to the oriented edge e_i is y_i for each $i \in \{1, 2, \ldots, g\}$.

Lemma 1.2. *For each basis $Y = \{y_1, y_2, \ldots, y_g\}$ of F, there exists a* 1*-vertex spine of V with the vertex at v_0 that is compatible with Y.*

Proof. Let φ be the automorphism of F that maps x_i to y_i for each $i \in \{1, 2, \ldots, g\}$. By Nielsen [31], the map φ can be expressed as the composition $\varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$, where each φ_i is one of the four *elementary Nielsen transformations*. We refer the reader to Magnus-Karrass-Solitar [28] for details on the elementary Nielsen transformations. For each elementary Nielsen transformation φ_i, there exists a homeomorphism f_i of V that fixes v_0 such that $f_i(\Gamma_0)$ is compatible with the basis $\{\varphi_i(x_1), \varphi_i(x_2), \ldots, \varphi_i(x_g)\}$. It follows that $f_n \circ \cdots \circ f_2 \circ f_1(\Gamma_0)$ is a required 1-vertex spine of V. □

Let M be a compact orientable irreducible 3-manifold with non-empty boundary with a base point v. We say that M satisfies the *strong bounded Kneser conjecture* (*SBKC*) if whenever we have subgroups G_1, G_2 of $\pi_1(M, v)$ with $G_1 \cap G_2 = 1$, $\pi_1(M, v) = G_1 * G_2$ and $G_i \not\cong 1$ $(i = 1, 2)$, there exists a properly embedded disk D in M containing v such that D separates M into two components M_1 and M_2 with $\iota_{i*}(\pi_1(M_i, v)) = G_i$ $(i = 1, 2)$, where $\iota_i : M_i \hookrightarrow M$ is the natural embedding. As we will see in the remark after the proof of Lemma 1.4, there exists a 3-manifold that does not satisfy SBKC. It follows directly from Lemma 1.2 that a genus g handlebody V satisfies the SBKC. In fact, for each decomposition $\pi_1(V, v_0) = G_1 * G_2$, we have a 1-vertex spine Γ of V having the vertex at v_0 that is compatible with the basis $\{y_1, y_2, \ldots, y_g\}$, where $\{y_1, y_2, \ldots, y_{g_1}\}$ is a basis of G_1 and $\{y_{g_1+1}, y_{g_1+2}, \ldots, y_g\}$ is a basis of G_2. Using the spine Γ, we have the required disk D. We note that a sufficient condition for a manifold to satisfy the SBKC was given by Jaco [20] as follows.

Lemma 1.3 (Jaco [20])**.** *Let M be a compact orientable irreducible* 3*-manifold with non-empty, connected boundary. Suppose that $\pi_1(M)$ is freely reduced, that is, if whenever*

we have a decomposition $G = G_1 * G_2$ then none of G_1 and G_2 is a free group. Then M satisfies the SBKC.

Lemma 1.4. *Let M be a compact irreducible 3-manifold with non-empty boundary. Let K be an oriented knot in the interior of M. If K binds $\pi_1(M)$, then K fills up M. Moreover, the converse is true when M satisfies the SBKC.*

Proof. Suppose that K does not fill up M. Then there exists an incompressible sphere or a compression disk D for ∂M in $M \setminus K'$, where K' is a knot with $K \overset{M}{\sim} K'$. By the same argument as in the second half of the proof of Lemma 1.1, using K' instead of K in the proof, we can show that K does not bind $\pi_1(M)$.

Next, suppose that M satisfies the SBKC, and K does not bind $\pi_1(M)$. We fix an orientation and a base point v of K. There exist subgroups G_1, G_2 of $\pi_1(M, v)$ with $G_1 \cap G_2 = 1$, $\pi_1(M, v) = G_1 * G_2$, $G_2 \not\cong 1$, and $[K] \in G_1$. If $G_1 = 1$, then K is contractible and thus we are done. Suppose that $G_1 \not\cong 1$. Then by the SBKC, there exists a properly embedded disk D in M containing v such that D separates M into two components M_1 and M_2 with $\iota_{i*}(\pi_1(M_i, v)) = G_i$ $(i \in \{1, 2\})$, where $\iota_i : M_i \hookrightarrow M$ is the natural embedding. We may assume that K is moved by a homotopy fixing v so that $|K \cap D|$ is minimal. If $|K \cap D| = 0$, we are done. Suppose that $|K \cap D| > 0$. Then $[K]$ can be decomposed into a product $x_1 \cdot x_2 \cdots x_r$, where x_i is in G_1 or G_2, and x_i, x_{i+1} do not lie in one of G_1 and G_2 at the same time. We note that $r > 1$. Since $[K] \subset G_1$, at least one, say x_{i_0}, of $x_1, x_2, \ldots, x_r$ is trivial. Then moving a neighborhood of the subarc of K corresponding to x_{i_0} by a homotopy, we can reduce $|K \cap D|$. This contradicts the minimality of $|K \cap D|$. This completes the proof. □

We remark that the converse of Lemma 1.4 is not true. This can be seen as follows. Let Σ be a closed orientable surface of genus at least one. Let M be a 3-manifold obtained by attaching a 1-handle H to $\Sigma \times [0, 1]$ so as to connect $D \times \{0\}$ and $D \times \{1\}$ and that the resulting manifold M is orientable, where D is a disk in Σ. See Figure 2.

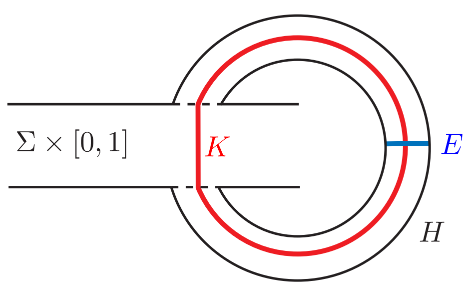

FIGURE 2. The manifold M.

Clearly, M is compact, orientable and irreducible. Let $K \subset M$ be the knot obtained by extending the core of H along a vertical arc $\{*\} \times [0, 1]$ in $\Sigma \times [0, 1]$. We fix a base point v in K and an orientation of K. Then the fundamental group $\pi_1(M, v)$ can be naturally identified with $\pi_1(\Sigma) * \mathbb{Z}$, and under this identification $[K]$ is contained in the factor $\mathbb{Z}$. This implies that K does not bind $\pi_1(M)$. On the contrary, it is easy to see that the co-core E of the 1-handle H is the unique compression disk for ∂M up to isotopy. The

algebraic intersection number of K and E is ± 1 after giving an orientation of E. This implies that after deforming K by any homotopy in M, K intersects E, whence K fills up M. We note that M does not satisfy SBKC.

Lemma 1.5. *Let V be a handlebody. Then there exists a knot in the interior of V that fills up V.*

Proof. Let K be a simple closed curve in ∂V such that $\partial V \setminus K$ is incompressible in V. Such a simple closed curve does exist. In fact, a simple closed curve shown in Figure 3 satisfies this condition (see for instance Wu [47, Section 1]).

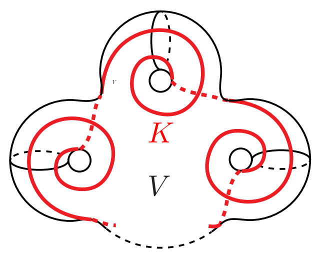

FIGURE 3. The surface $\partial V \setminus K$ is incompressible in V.

Then by Lemma 1.1 K binds $\pi_1(V)$. It follows from Lemma 1.4 that a knot obtained by moving K by an isotopy to lie in the interior of V fills up V. □

2. Knots filling up a 3-subspace of the 3-sphere

Let V be a handlebody. A (possibly disconnected) subgraph of a spine of V is called a *subspine* if it does not contain a contractible component. A *compression body* W is the complement of an open regular neighborhood of a (possibly empty) subspine Γ of a handlebody V. The component $\partial_+ W = \partial V$ is called the *exterior boundary* of W, and $\partial_- W = \partial W \setminus \partial_+ W = \partial \mathrm{Nbd}(\Gamma)$ is called the *interior boundary* of W. We remark that the interior boundary is incompressible in W, see Bonahon [3].

For a compression body W, a *spine* is defined to be a graph Γ embedded in W so that

(1) $\Gamma \cap \partial W = \Gamma \cap \partial_- W$ consists only of vertices of valence one; and
(2) W collapses onto $\Gamma \cup \partial_- W$.

We note that this is a generalization of a spine of a handlebody. We also note that if V is a handlebody and Γ is a subspine of $\hat{\Gamma}$ of V such that $W \cong V \setminus \mathrm{IntNbd}(\Gamma; V)$, then $\hat{\Gamma} \setminus \mathrm{IntNbd}(\Gamma; V)$ is a spine of W. As a generalization of the case of handlebodies, a 1-*vertex spine* of a compression body W is defined to be a (possibly empty) connected spine Γ such that

(1) Γ is homeomorphic to the empty set, an interval, a circle, or a graph with a single vertex of valence at least 3;
(2) Γ intersects each component of $\partial_- W$ in a single univalent vertex; and
(3) Γ has no univalent vertices in the interior of W.

If Γ is an interval or a circle, we regard that Γ contains a unique vertex of valence 2. For a 1-vertex spine of a compression body W, The spines shown in Figure 4 (i)-(iii) are

1-vertex spines while the one shown in Figure 4 (iv) is not so because it has a univalent vertex in the interior of W.

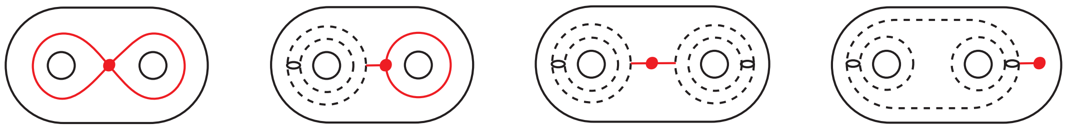

FIGURE 4

we call a vertex of valence at least 2 the *interior vertex*. We note that every 1-vertex spine has a unique interior vertex. This is the reason why it is named so.

Let W be a compression body. Suppose that $\partial_- W$ consists of n closed surfaces $\Sigma_1,\ \Sigma_2, \ldots, \Sigma_n$. A (possibly empty) set $\mathcal{D} = \{D_1,\ D_2, \ldots, D_m,\ E_{\Sigma_1},\ E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ of pairwise disjoint compression disks for $\partial_+ W$ is called a *cut-system* for W if

(1) each disk E_{Σ_i} separates from W a component that is homeomorphic to $\Sigma_i \times [0,1]$ and contains Σ_i;
(2) W cut off by $E_{\Sigma_1} \cup E_{\Sigma_2} \cup \cdots \cup E_{\Sigma_n}$ has at most one handlebody component V; and
(3) $D_1 \cup D_2 \cup \cdots \cup D_m$ cuts off V into a single 3-ball. See Figure 5.

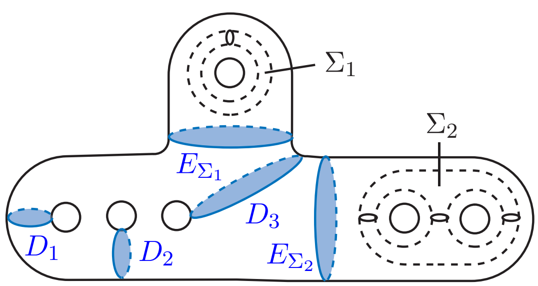

FIGURE 5. A cut system.

We note that if $W = \Sigma \times [0,1]$, where Σ is a closed orientable surface, then $m = n = 0$. If W is a handlebody, then $n = 0$ and m is its genus.

By virtue of the Poincaré-Lefschez duality, we have a one-to-one correspondence between the 1-vertex spines and cut-systems of a compression body W modulo isotopy. The correspondence can be described as follows. The 1-vertex spine Γ *dual to* a given cut-system $\mathcal{D}$ for a compression body W is obtained by regarding a regular neighborhood of each disk D in $\mathcal{D}$ as a 1-handle with D as the cocore, and then extending the core arcs of the 1-handles in each component W_0 of the exterior of the union of the disks in $\mathcal{D}$ in such a way that

(1) if W_0 is a 3-ball, then the extension is given by radial arcs; and
(2) if W_0 is the product of a closed surface with an interval, then the extension is given by an vertical arc.

By conversing the construction, we get the cut-system *dual to* a 1-vertex spine of W.

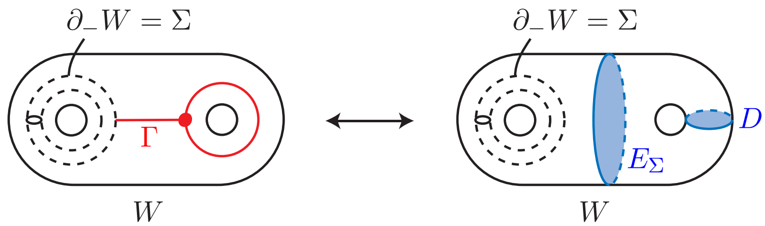

FIGURE 6. The Poincaré-Lefschez duality.

Let V be a handlebody of genus g and Γ be a subspine of V. Assume that each component of Γ is a rose. A *cut-system* for the pair (V, Γ) is a cut-system for V dual to a spine $\hat{\Gamma}$, where $\hat{\Gamma}$ is obtained by contracting a maximal subtree of a spine of V containing Γ' as a subgraph. See Figure 7.

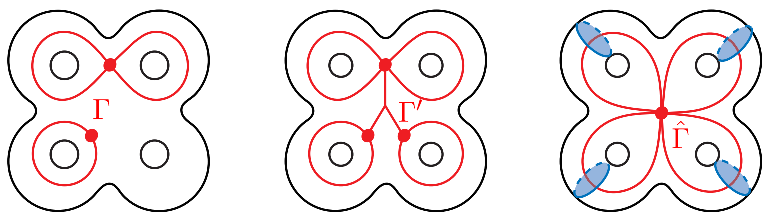

FIGURE 7. A cut-system for (V, Γ) is a cut-system for V dual to a spine $\hat{\Gamma}$.

Lemma 2.1. *Let W be a compression body. Let D be a compression disk for $\partial_+ W$. Then there exists a cut-system for W disjoint from D.*

Proof. We may identify W with a genus g handlebody V with an open regular neighborhood of a subspine Γ removed. Further, we may assume that each component of Γ is a rose. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be the components of Γ. Choose a cut-system $\{D_1, D_2 \ldots, D_g\}$ for the pair (V, Γ) so that $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$ is minimal among all cut-systems for (V, Γ).

Suppose for a contradiction that $D \cap (D_1 \cup D_2 \cup \cdots \cup D_g) \neq \emptyset$. Choose an outermost subdisk δ of D cut off by $D_1 \cup D_2 \cup \cdots \cup D_g$. We may assume that $\delta \cap D_1 \neq \emptyset$. Let D_1' and D_1'' be the disks obtained from D_1 by surgery along δ. Then exactly one of $\{D_1', D_2 \ldots, D_g\}$ and $\{D_1'', D_2 \ldots, D_g\}$, say $\{D_1', D_2 \ldots, D_g\}$, is a cut-system for the handlebody V. We note that D_1'' separates the handlebody V cut off by $D_2 \cup D_3 \cup \cdots \cup D_g$. If D_1 does not intersect Γ, then it follows that $\{D_1', D_2 \ldots, D_g\}$ is a cut-system for the pair (V, Γ) with $|D \cap (D_1' \cup D_2 \cup \cdots \cup D_g)| < |D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. This contradicts the minimality of $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. Suppose that D_1 intersects Γ. If D_1'' intersects Γ, then D_1'' can not separate the handlebody V cut off by $D_2 \cup D_3 \cup \cdots \cup D_g$. This is a contradiction. Thus D_1' intersects Γ. This implies that $\{D_1', D_2 \ldots, D_g\}$ is a cut-system for the pair (V, Γ). This contradicts again, the minimality of $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. Therefore we have $D \cap (D_1 \cup D_2 \cup \cdots \cup D_g) = \emptyset$ and $D \cap \Gamma = \emptyset$.

From now on, we assume that each of $D_1, D_2, \ldots, D_m$ does not intersect Γ while each of $D_{m+1}, D_{m+2}, \ldots, D_g$ does so. Let B the 3-ball obtained by cutting V along $D_1 \cup D_2 \cup \cdots \cup D_g$. Then $B \cap \Gamma_i$ is a cone on an even number of points. We note that

D is a separating disk in B disjoint from the cones $B \cap \Gamma$. For each $i \in \{1, 2, \ldots, m\}$ let $D_i^{\pm}$ be disks on the boundary of B coming from D_i. Then there exists a set $\{E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ of mutually disjoint disks properly embedded in B such that

(1) $E_{\Sigma_1} \cup E_{\Sigma_2} \cup \cdots \cup E_{\Sigma_n}$ is disjoint from $\Gamma \cup D \cup D_1^{\pm} \cup D_2^{\pm} \cup \cdots \cup D_g^{\pm}$; and
(2) E_{Σ_i} separates from B a 3-ball B_i such that $B_i \cap \Gamma = B \cap \Gamma_i$ and $B_i \cap (D_1^{\pm} \cup D_2^{\pm} \cup \cdots \cup D_m^{\pm}) = \emptyset$.

Now the set $\{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ is a required cut-system for W. □

Let M be an irreducible, compact, connected, orientable 3-manifold with connected boundary. Following Bonahon [3], a *characteristic compression body* W of M is defined to be a compression body embedded in M so that

(1) $\partial_+ W = \partial M$; and
(2) The closure of $M \setminus W$ is boundary-irreducible.

We remark that, for a given characteristic compression body W of M, by the irreducibility of M, every compression disk for ∂M can be moved by an isotopy to lie in W.

Theorem 2.2 (Bonahon [3]). *An irreducible, compact, connected, orientable* 3*-manifold with connected boundary has a unique* (*up to isotopy*) *characteristic compression body.*

Lemma 2.3. *Let M be a compact, connected, orientable* 3*-manifold with connected boundary. Let W be a compression body in M such that $\partial M = \partial_+ W$. Let K be a knot in the interior of W. If K fills up M, then K fills up W. Further, when M is irreducible and W is the characteristic compression body, then K fills up M if and only if K fills up W.*

Proof. Since any knot K' in the interior of W with $K \overset{W}{\sim} K'$ satisfies $K \overset{M}{\sim} K'$, it follows immediately from the definition that if K fills up M, then K fills up W.

Suppose M is irreducible, W is the characteristic compression body, and K is a knot in W that fills up W. We will show that K fills up M. If M is a handlebody, then we have $M = W$ and there is nothing to prove. Suppose that M is not a handlebody. Then M can be decomposed as $M = W \cup X$, where $W \cap X = \partial_- W = \partial X$ and X is the union of boundary-irreducible 3-manifolds. The interior boundary $\partial_- W$ consists of a finite number of closed surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ of genus at least 1. Let g_i be the genus of Σ_i ($i \in \{1, 2, \ldots, n\}$). We recall that each Σ_i is incompressible in M. Suppose for a contradiction that there exists a knot K' in the interior of M with $K \overset{M}{\sim} K'$ such that ∂M is compressible in $M \setminus K'$. Let D be a compression disk for ∂M in $M \setminus K'$. We may assume that D is contained in W.

Suppose first that D does not separate W. By Lemma 2.1, there exists a cut-system for W disjoint from D. By replacing a suitable disk in the system with D, we obtain a cut-system $\mathcal{D} = \{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ where $D = D_1$. Let Γ be the 1-vertex spine of W dual to $\mathcal{D}$. Fix a presentation of the fundamental group of each surface Σ_i as: $\pi_1(\Sigma_i) = \langle a_{i,j}, b_{i,j}\ (j \in \{1, 2, \ldots, g_i\}) \mid \prod_{j=1}^{g_i} [a_{i,j}, b_{i,j}] \rangle$, where we take the base point at $\Gamma \cap \Sigma_i$.

Let v_0 be the interior vertex of Γ. Let V be the unique component of W cut off by the union of disks in $\mathcal{D}$ that is homeomorphic to a handlebody. We fix a generating set $\{x_1, x_2, \ldots, x_m\}$ of $\pi_1(V, v_0)$ so that an element x_i is defined by the loop in Γ dual to

D_i. Then by the Seifert-van Kampen Theorem, $\pi_1(W, v_0)$ is generated by x_i's, $a_{i,j}$'s and $b_{i,j}$'s. Set

$$G = \{{x_i}^{\pm 1} \mid i \in \{1, 2, \ldots, m\}\} \cup \{{a_{i,j}}^{\pm 1}, {b_{i,j}}^{\pm 1} \ (j \in \{1, 2, \ldots, g_i\}) \mid i \in \{1, 2, \ldots, n\}\}.$$

Let H_1, $H_2, \ldots, H_l$ be 1-handles in X attached to $\partial_- W$ so that the closure of $M \setminus (W \cup H_1 \cup H_2 \cup \cdots \cup H_l)$ is the union of handlebodies. Let h_1, $h_2, \ldots, h_l$ be the element of $\pi_1(M, v_0)$ corresponding to the core of the 1-handles H_1, $H_2, \ldots, H_l$, respectively. We set

$$\hat{G} = G \cup \{{h_i}^{\pm 1} \mid i \in \{1, 2, \ldots, l\}\}.$$

We note that the elements $\hat{G}$ generates the group $\pi_1(M, v_0)$. In other words, any element of $\pi_1(M, v_0)$ can be represented by a word on $\hat{G}$.

Since each Σ_i is incompressible in M, $\pi_1(W, v_0)$ is a subgroup of $\pi_1(M, v_0)$. Consider the conjugation class $c_{\pi_1(W,v_0)}(K)$. Since K fills up W, every word w on G representing an element of $c_{\pi_1(W,v_0)}(K)$ contains $x_1^{\pm 1}$.

By the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. Let u be a word on $\hat{G}$ such that $u^{-1}wu$ represent the same element as w' in $\pi_1(M, v_0)$. Let $\varphi : \pi_1(M, v_0) \to \pi_1(W, v_0)$ be the epimorphism obtained by adding relations $h_i = 1$ for each $i \in \{1, 2, \ldots, l\}$. For a word v, we denote by $\varphi(v)$ the word on G obtained from v by replacing each ${h_i}^{\pm}$ in the word with $\emptyset$. Then $\varphi(u^{-1}wu) = \varphi(u)^{-1}w\varphi(u)$ represents an element contained in $c_{\pi_1(W,v_0)}(K)$. It follows that $\varphi(w')$ is a word on $G \setminus \{{x_1}^{\pm}\}$ representing an element of $c_{\pi_1(W,v_0)}(K)$. This is a contradiction.

Next, suppose that D separates W into two components W_1 and W_2. By Lemma 2.1, there exists a cut-system $\mathcal{D} = \{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ for W disjoint from D. Without loss of generality we can assume that the disks of $\mathcal{D}$ contained in W_1 is $\{D_1, D_2, \ldots, D_{m_1}, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_{n_1}}\}$, where $m_1 \in \{1, 2, \ldots, m\}$ and $n_1 \in \{0, 1, \ldots, m\}$. Here we put $n_1 = 0$ if none of $\{E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ is contained in W_1.

Let Γ be the 1-vertex spine of W dual to $\mathcal{D}$. Using the spine Γ, fix generating sets

$$G = \{{x_i}^{\pm 1} \mid i \in \{1, 2, \ldots, m\}\} \cup \{{a_{i,j}}^{\pm 1}, {b_{i,j}}^{\pm 1} \mid i \in \{1, 2, \ldots, n\},\ j \in \{1, 2, \ldots, g_i\}\}$$

of $\pi_1(W, v_0)$ and

$$\hat{G} = G \cup \{{h_i}^{\pm 1} \mid i \in \{1, 2, \ldots, l\}\}.$$

of $\pi_1(M, v_0)$ and an epimorphism $\varphi : \pi_1(M, v_0) \to \pi_1(W, v_0)$ as in the above argument.

If $m_1 \neq m$, then by the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ or $\hat{G} \setminus \{{x_m}^{\pm 1}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. By the same argument as in the case where D is non-separating, this is a contradiction. If $m_1 = m$, then $n_1 \neq n$. Hence by the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ or $\hat{G} \setminus \{{a_{n,j}}^{\pm 1}, {b_{n,j}}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. It follows that $\varphi(w')$ is a word on $G \setminus \{{x_1}^{\pm 1}\}$ or $G \setminus \{{a_{n,j}}^{\pm 1}, {b_{n,j}}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$ representing an element of $c_{\pi_1(W,v_0)}(K)$. However, this is again a contradiction because of the fact that K fills up W implies that every word on G representing an element of $c_{\pi_1(W,v_0)}(K)$ contains both one of $x_1^{\pm 1}$ and one of $\{{a_{n,j}}^{\pm 1}, {b_{n,j}}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$. This completes the proof. $\square$

Theorem 2.4. *Let M be a compact, connected, proper 3-submanifold of S^3 with connected boundary. Then there exists a knot K in the interior of M that fills up M. Moreover, such a knot K can be taken to lie in* $\mathrm{Nbd}(\partial M; M)$.

Proof. If M is a handlebody, the assertion follows from Lemma 1.5. Suppose that M is not a handlebody. Let W be the characteristic compression body of M. We may identify W with the complement of an open regular neighborhood of a subspine Γ of a handlebody V. Let K be a knot in the interior of V that fills up V. Since K can be taken not to intersect a spine of V containing Γ as a subgraph, We may assume that K lies in a collar neighborhood of $\partial_+ W = \partial M$. By Lemma 2.3, K fills up W. Thus, again by Lemma 2.3, K fills up M. This completes the proof. □

3. Transient knots in a subspace of the 3-sphere

Let M be a compact, connected, proper 3-submanifold of S^3. A knot K in $M \subset S^3$ is said to be *transient in M* if K can be deformed by a homotopy in M to be the trivial knot in S^3. Otherwise, K is said to be *persistent in M*.

Example. The knot K_1 described on the left-hand side in Figure 8 is transient in the handlebody V_1 in S^3 while the knot K_2 described on the right-hand side is persistent in V_2.

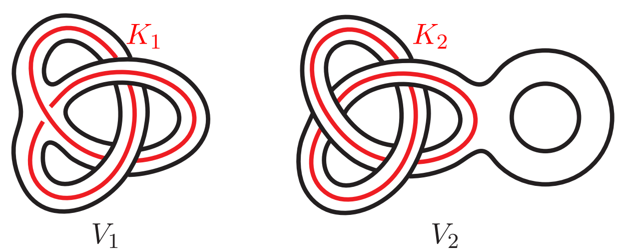

FIGURE 8. The knot K_1 is transient in V_1 while K_2 is persistent in V_2.

The following lemma follows straightforwardly from the definition.

Lemma 3.1. *Let M be a compact, connected, proper 3-submanifold of S^3 and let N be a compact, connected 3-submanifold of M. If a knot K in N is persistent in M, then so is in N.*

A compact, connected, proper 3-submanifold M of S^3 is said to be *unknotted* if the exterior $E(M)$ is a disjoint union of handlebodies. Otherwise M is said to be *knotted*. We recall that a theorem of Fox [13] says that any compact, connected, proper 3-submanifold of S^3 can be re-embedded in S^3 in such a way that its image is unknotted. See Scharlemann-Thompson [41] and Ozawa-Shimokawa [33] for certain generalizations and refinements of Fox's theorem.

Remark. As mentioned in the Introduction, M usually admits many non-isotopic embeddings into S^3 with the unknotted image. The uniqueness holds for a handlebody by Waldhausen [46]. Here the uniqueness is up to isotopy for subsets of S^3, where we recall that two subsets M_1 and M_2 of S^3 is isotopic if and only if there exists an

orientation-preserving homeomorphism f of S^3 carrying M_1 onto M_2. If we consider isotopies not between the embedded subsets but between embeddings, it is far from being unique even for a handlebody. This can be explained under a general setting as follows: Let M be a compact, connected 3-submanifold M that can be embedded in S^3. Then its mapping class group $\mathcal{MCG}_+(M)$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of M. We fix an embedding $\iota_0 : M \to S^3$. Let $\mathcal{G}_{\iota_0(M)} = \mathcal{MCG}_+(S^3, \iota_0(M))$ be the mapping class group of the pair $(S^3, \iota_0(M))$, that is, the group of isotopy classes of orientation-preserving homeomorphisms of S^3 that preserve $\iota_0(M)$. See Koda [23] for details of this group when M is a knotted handlebody. We can define an injective homomorphism $\iota_0^* : \mathcal{G}_{\iota_0(M)} \hookrightarrow \mathcal{MCG}_+(M)$ by assigning to each homeomorphism $\varphi \in \mathcal{G}_{\iota_0(M)}$ a unique element f of $\mathcal{MCG}_+(M)$ satisfying $\varphi \circ \iota_0 = \iota_0 \circ f$. Then the set of embeddings of M into S^3 with the same image up to isotopy can be identified with the right cosets $\iota_0^*(\mathcal{G}_{\iota_0(M)}) \backslash \mathcal{MCG}_+(M)$, where the identification is given by assigning to $f \in \mathcal{MCG}_+(M)$ the embedding $\iota_0 \circ f : M \to S^3$. When M is a handlebody of genus at least two, it is clear that this is an infinite set. We note that when $\iota_0(M)$ is an unknotted handlebody of genus two, the group $\mathcal{G}_{\iota_0(M)}$ is called the genus two Goeritz group of S^3 and studied in Goeritz [14], Scharlemann [39], Akbas [1] and Cho [7].

Let K be a knot in M. Let f is contained in the coset $\iota_0^*(\mathcal{G}_{\iota_0(M)})\mathrm{id}_M$. By the above observation and the definition of the persistency of knots in $M \subset S^3$, it follows immediately that $\iota_0 \circ f(K)$ is persistent in M if and only if so is K. We note that if f is not contained in the coset $\iota_0^*(\mathcal{G}_{\iota_0(M)})\mathrm{id}_M$, then the knot $\iota_0 \circ f(K)$ is not necessarily persistent in M even if K is persistent in M. See Figure 9.

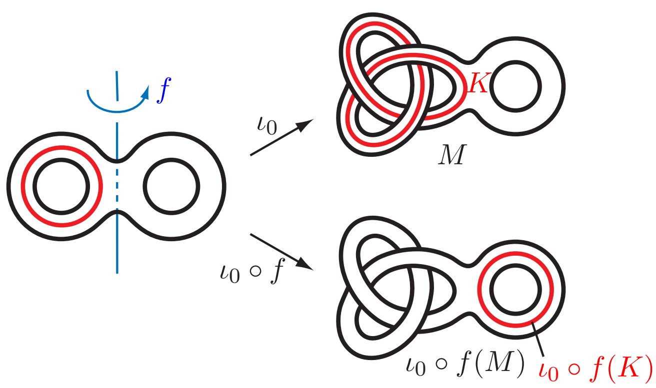

Figure 9. Persistency is an extrinsic property.

Be that as it may, we discuss in this paper extrinsic properties of knots embedded submanifold of S^3, not intrinsic one.

Theorem 3.2. *Let M be a compact, connected, proper 3-submanifold of S^3. Then every knot in M is transient if and only if M is unknotted.*

Proof. Suppose first that M is unknotted, i.e. $M = S^3 \setminus \mathrm{Int}\,\mathrm{Nbd}(\Gamma)$, where Γ is a graph embedded in M. Let K be an arbitrary knot in M. Considering a diagram of the spatial graph $K \cup \Gamma$, we easily see that K can be converted into the trivial knot in S^3 by a finite number of crossing changes of K itself. This implies that K is transient in M.

Next suppose that M is knotted. Then there exists a component N of the exterior of M that is not a handlebody. Let W be the characteristic compression body of N. We note that if N is boundary-irreducible, then W is a collar neighborhood of ∂N in N. Since W is not a handlebody, we can take a non-empty component Σ of $\partial_- W$. Then Σ separates S^3 into two components X and Y so that X is boundary-irreducible and Y contains $M \cup W$. See Figure 10.

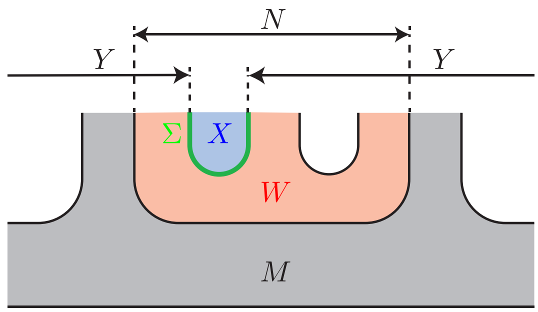

FIGURE 10. The configurations of M, N, W, Σ, X and Y.

By Theorem 2.4, there exists a knot K lying in $\mathrm{Nbd}(\partial Y; Y)$ that fills up Y. In particular K lies in W. Thus by an isotopy we can move K to lie within M. Let $K' \subset M$ be an arbitrary knot with $K \overset{M}{\sim} K'$. Since K fills up Y, Σ is incompressible in $Y \setminus K'$. Thus Σ is incompressible in $S^3 \setminus K'$. This implies that K' is not the trivial knot in S^3. Therefore K is persistent in M. $\square$

Remark. Let M be a compact, connected, knotted, proper 3-submanifold of S^3. In the proof of Theorem 3.2, we explained how to obtain a knot in M that is persistent in M. In the process, some readers may have guessed that if a knot $K \subset M$ filled up M, then K would already be persistent. If so, the process to consider the characteristic compression body of a non-handlebody component of the exterior in the proof is not necessary. However, the guess is not true in fact. Let K be the knot in the genus two knotted handlebody $V \subset S^3$ as shown in Figure 11.

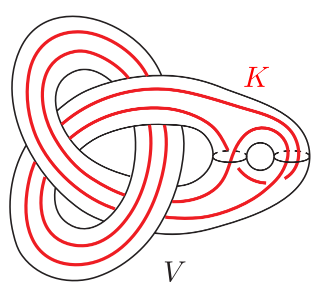

FIGURE 11. The knot K fills up V whereas K is transient in V.

Then we see that K fills up V by the same reason as in the proof of Lemma 1.5 (see also Section 6 (2)) whereas K is apparently transient in V.

4. Construction of persistent knots

4.1. **Persistent laminations and persistent knots.** Let M be a compact, connected, proper 3-submanifold of S^3 whose exterior consists of boundary-irreducible 3-manifolds. It is easy to see that every knot filling up M is persistent in M. Indeed, if a knot K in M fills up M, then each component of ∂M will be an incompressible surface in the exterior of any knot K' homotopic to K in V, hence K' is not the trivial knot in S^3. However, the converse is false in general as we see in the following proposition:

Proposition 4.1. *There exists a genus two handlebody V embedded in S^3 with the boundary-irreducible exterior such that there exists a knot $K \subset V$ which is persistent in V, and which does not fill up V.*

Proof. Let V be the genus two handlebody in S^3 and K be the knot in V as shown in Figure 12.

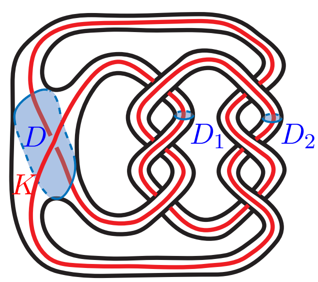

Figure 12. A handlebody V in S^3 with the boundary-irreducible exterior such that there exists a knot $K \subset V$ which is persistent in V, and which does not fill up V.

We note that the handlebody V is the exterior of the Brittenham's branched surface [5] constructed from a disk spanning the trivial knot in S^3. In particular, the exterior of V is boundary-irreducible. We note that K does not fill up V since there exists a compression disk D for ∂V in $V \setminus K$ as shown in the figure.

In the following, we will show that K is persistent in V. As illustrated in the figure, there are meridian disks D_1, D_2 of V each of which intersects K once and transversely. Let K' be any knot homotopic to K in V. Then K' intersects each of D_1 and D_2 at least once. By Hirasawa-Kobayashi [17] or Lee-Oh [25] that generalizes the result of Brittenham [5], in the exterior of V there exists a *persistent lamination*, that is, an essential lamination that remains essential after performing any non-trivial Dehn surgeries along K'. This implies that K' is not the trivial knot, thus K is persistent in V. □

4.2. **Accidental surfaces and persistent knots.** A closed essential surface Σ in the exterior of a knot K in the 3-sphere is called an *accidental surface* if there exists an annulus A, called an *accidental annulus*, embedded in the exterior $E(K)$ such that

- the interior of A does not interect $\Sigma \cup \partial E(K)$,

- $A \cap \Sigma \neq \emptyset$ and $A \cap \partial E(K) \neq \emptyset$, and these are essential simple closed curves in Σ and $\partial E(K)$, respectively.

In Ichihara-Ozawa [18] it is shown that for each accidental surface in the exterior of a knot in S^3, the boundary curves of accidental annuli determine the unique slope on the boundary of a regular neighborhood of the knot. This slope is called an *accidental slope* for Σ. By Culler-Gordon-Luecke-Shalen [11], an accidental slope is either meridional or integral.

Proposition 4.2. *Let M be a compact, connected, proper 3-submanifold of S^3 with connected boundary such that the exterior of M is boundary irreducible. Let K be a knot in M such that ∂M is incompressible in $M \setminus K$. If ∂M is an accidental surface with integral accidental slope in the exterior of K, then K is persistent in the submanifold M of S^3 bounded by Σ and containing K.*

Proof. Let $A \subset M$ be an accidental annulus connecting K and a simple closed curve in ∂M. Using this annulus, we move K to a knot K^* lying in ∂M by an isotopy. Since ∂M is incompressible in $E(K)$, $\partial M \setminus K^*$ is incompressible in M. Thus by Lemma 1.1 K^* binds $\pi_1(M)$, so does K. By Lemma 1.4, K fills up M. Let $K' \subset M$ be an arbitrary knot lying in the interior of M with $K \overset{M}{\sim} K'$. Since K fills up M, ∂M is incompressible in $M \setminus K'$. Thus ∂M is incompressible in $S^3 \setminus K'$. This implies that K' is not the trivial knot in S^3. Therefore K is persistent in M. □

5. Transient number of knots

Let K be a knot in S^3. A *crossing move* on a knot K is the operation of passing one strand of K through another. The *unknotting number* $u(K)$ of K, which was first defined by Wendt [45], is then the minimal number of crossing moves required to convert the knot into the trivial knot. We note that for each crossing move, we can associate a simple arc α in S^3 such that $\alpha \cap K = \partial\alpha$ and the crossing move is performed in $\mathrm{Nbd}(\alpha)$.

An *unknotting tunnel system* for K is a set $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of mutually disjoint simple arcs in S^3 such that $\gamma_i \cap K = \partial\gamma_i$ for each $i \in \{1, 2, \dots, n\}$ and the exterior of the union $K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ is a handlebody. The *tunnel number* $t(K)$ of K, which was first defined by Clark [9], is the minimal number of arcs in any of unknotting tunnel systems for K.

We introduce a new invariant for a knot in the 3-sphere strongly related to the above two classical invariants. We define *transient system* for K to be a set $\{\tau_1, \tau_2, \dots, \tau_n\}$ of mutually disjoint simple arcs in S^3 such that $\tau_i \cap K = \partial\tau_i$ for each $i \in \{1, 2, \dots, n\}$ and K is transient in $\mathrm{Nbd}(K \cup \tau_1 \cup \tau_2 \cup \cdots \cup \tau_n)$. The *transient number* $tr(K)$ of K is defined to be the minimal number of arcs in any of transient systems for K.

Proposition 5.1. *Let K be a knot in S^3. Then we have $tr(K) \leqslant u(K)$ and $tr(K) \leqslant t(K)$.*

Proof. Suppose that $u(K) = m$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be a set of mutually disjoint simple arcs associated to m crossing moves that convert K into the trivial knot. Then K is transient in the handlebody $\mathrm{Nbd}(K \cup \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m)$. In other words, $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is a transient tunnel system for K. This implies that $tr(K) \leqslant m$.

Suppose that $t(K) = n$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be an unknotting tunnel system for K. Since the handlebody $\mathrm{Nbd}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ is unknotted, K is transient in $\mathrm{Nbd}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ by Theorem 3.2. This implies that $tr(K) \leqslant n$. □

Proposition 5.2. *There exists a knot K in S^3 such that $tr(K) = 1$ and $u(K) = t(K) = 2$.*

Proof. Let K be the satelite knot of the figure eight knot shown in Figure 13.

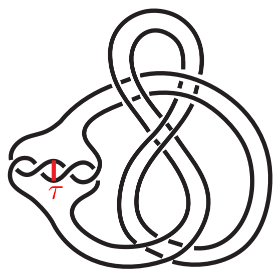

FIGURE 13. A knot K with $tr(K) = 1$ and $u(K) = t(K) = 2$.

Clearly, the genus of K is one. The transient number of K is one because K admits a transient tunnel as shown in the figure. In Kobayashi [21] and Scharlemann-Thompson [40], it is proved that the only non-simple knots of genus one and unknotting number one are the doubled knots. It follows that the unknotting number of K is at least two. It is then straightforward to see that the unknotting number is exactly two.

It is proved by Morimoto-Sakuma [29] that the only non-simple knots having unknotting tunnels are certain satellites of torus knots. it follows that the tunnel number of K is at least two. It is then straightforward to see that the tunnel number is exactly two. □

6. Concluding remarks

(1) Let M be a compact, connected, proper 3-submanifold of S^3. Let K be a knot in the interior of M. In the earlier sections, we have introduced various homotopic properties of knots in M. We summarize their relations. We say that K is *accidental* in M if K can be moved to a knot K' in ∂M by a homotopy in M so that $\partial M \setminus K'$ is incompressible in M. Then we have the following:
   (a) If K is accidental, then K binds $\pi_1(M)$ (c.f. Lemma 1.1).
   (b) If K binds $\pi_1(M)$, then K fills up M (c.f. Lemma 1.4).
   (c) By (1a) and (1b), if K is accidental, then K fills up M.

   The converse of each of them is false. To see this, suppose that M is the exterior of a non-trivial knot in S^3. We note that $\pi_1(M)$ is freely indecomposable by Kneser Conjecture. Let K be a knot in M that can not be moved by any homotopy in M to lie in ∂M. Such a knot K always exists by, for instance, Brin-Johannson-Scott [4]. This implies that K binds $\pi_1(M)$ whereas K is not accidental in M. In this example, we also see that K fills up M whereas K is

not accidental in M. The remark after the proof of Lemma 1.4 shows that the converse of Lemma 1.4 is false. However, the 3-manifold M introduced in the example is not embeddable in S^3. To have an counterexample of the converse of (1b), let Σ be a closed orientable surface of genus at least one. Let M be an orientable 3-manifold obtained by attaching a 1-handle to each component of $\partial(\Sigma \times [0,1])$. We note that M can be embedded in S^3. Let D_0 and D_1 be the co-core of the 1-handles. Then we can easily show as in the remark that there exists a knot K in M intersecting each of D_0 and D_1 once and transversely that fills up M whereas K does not bind $\pi_1(M)$. The relations of these the three intrinsic properties are shown on the left-hand side in Figure 14.

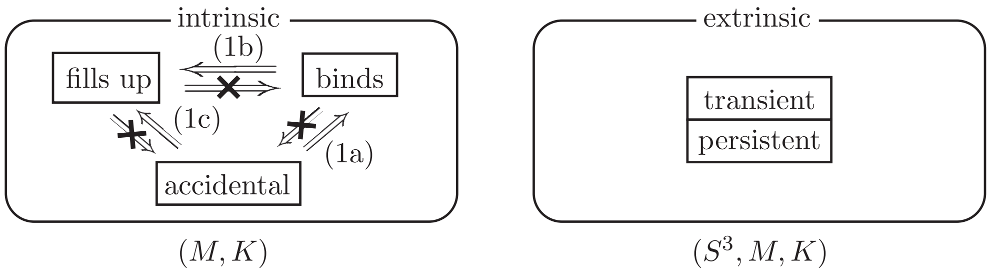

FIGURE 14. Corelation diagrams of extrinsic and intrinsic properties.

It is worth noting that to show that a given knot K in $M \subset S^3$ is persistent, we have used intrinsic property of K in a subset of S^3 containing M. See Theorem 3.2 and Propositions 4.1 and 4.2.

(2) Let F_g be a rank g free group. As mentioned in Section 1, an algorithm to detect whether a given element x of a free group F_g binds F_g is described by Stallings [43] using the combinatorics of its Whitehead graph. In fact, the following is proved:

Theorem 6.1 (Stallings [43]). *Let x be a cyclically reduced word on $X_g = \{x_1, x_2, \ldots, x_g\}$. If the Whitehead graph of x is connected and contains no cut vertex, then x binds F_g.*

For a simple closed curve in the boundary of a handlebody, this can be seen clearly as follows. Let x be an element of the rank g free group F_g. We identify F_g with the fundamental group of a genus g handlebody. In the case of $M = V_g$ in Lemma 1.1, which is actually Lyon [27, Corollary 1], we have seen that if x can be represented by an oriented simple closed curve K in ∂V_g, then x binds F_g if and only if $\partial V_g \setminus K$ is incompressible. On the other hand, Starr [44] (see also Wu [47, Theorem 1.2]) showed that $\partial V_g \setminus K$ is incompressible if and only if there is a complete meridian disk system $D_1, D_2, \ldots, D_g$ of V_g such that the planar graph with "fat" vertices obtained by cutting ∂V_g along $\bigcup_{i=1}^{g} D_i$ is connected and contains no cut vertex. This graph is actually nothing else but the Whitehead graph of x. (As explained in Stallings [43], we can obtain a gemetric interpretation of this for an arbitrary element of F_g if we consider the connected sum of g copies of $S^2 \times S^1$ instead of V_g.)

(3) Let M be a compact, connected, proper 3-submanifold of S^3. In the proofs of Theorem 3.2 and Propositions 4.1 and 4.2, we provided a way to show that a given knot $K \subset M$ is persistent in M. The key idea there is to find an essential surface (or lamination) in the exterior of M that is also essential in the exterior of any knot K' homotopic to K in M. As mentioned in the Introduction, another way to show the persistency was provided by Letscher [26] that uses what he calls the *persistent Alexander polynomial*.

Problem 1. Provide more methods for detecting whether a given knot $K \subset M$ is persistent.

(4) As we have summarized in Figure 14, the only extrinsic property of knots in a 3-subspace of S^3 we have considered in the present paper was transience (or persistency). Using this property, we have actually gotten an "if and only if" condition for a 3-subspace of S^3 being unknotted in Theorem 3.2. This is a first step for a relative version of Fox's program and a futher progress will be expected.

Problem 2. Consider other extrinsic properties of knots in $M \subset S^3$ to characterize how M is embedded in S^3.

We note that the case where M is a handlebody is already a very interesting problem. See e.g. Ishii [19], Koda [23] and Koda-Ozawa [24].

(5) As mentioned in the Introduction, the unknottedness of a 3-submanifold can be considered for an arbitrary closed, connected 3-manifold. Thus it is natural to ask the following:

Question 1. Generalize Theorem 3.2 for M in an arbitrary 3-manifold N.

(6) Finally, in Section 5, we defined an integer-valued invariant $tr(K)$, the transient number, for a knot K in S^3. This invariant is nice in the sense that it shows the knots of unknotting number 1 and those of tunnel number 1 from the same perspective as we have seen in Proposition 5.1. However, it remains unknown whether there exists a knot whose transient number is more than 1.

Question 2. The transient number $tr(K)$ can be arbitrary large?

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